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DIFFERENTIAL GAME WITH IMPULSIVE CONTROL OF ONE OF THE PLAYERS

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A differential game of encounter in which the minimizing player controls his acceleration in an impulsive manner, is considered. The opposing player controls his velocity, which is restricted in magnitude. The problem of optimal time distribution of a fixed number of the impulses is solved. The subject of this paper is related to those of [1—3].

1. Formulation of the problem. Let the motion of the players X and Y over a fixed time interval $[0, T]$ be determined by the following equations of motion and initial conditions:

$$x_1^{\cdot} = x_2, \qquad x_2^{\cdot} = u, \qquad y^{\cdot} = v \tag{1.1}$$
$$x_1(0) = x_1^{\circ}, \qquad x_2(0) = x_2^{\circ}, \qquad y(0) = y^{\circ}$$

The vectors x_1, x_2, u, y, v all have the same arbitrary dimensions. We fix n instants of time $t_i, i = 1, \ldots, n$ on the interval $[0, T]$, and

$$t_0 = 0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant t_{n+1} = T \tag{1.2}$$

The realizations of the controls $u(t)$ of the player X and $v(t)$ of the player Y are subjected to the following restrictions ($\delta(t)$ is the delta function):

$$u(t) = \sum_{k=1}^{n} u_k \delta(t - t_k), \qquad \sum_{k=1}^{n} |u_k| \leqslant Q, \quad Q > 0; \qquad |v(t)| \leqslant 1 \tag{1.3}$$

Thus the velocity of the player X undergoes a jump of magnitude u_k, at the instant t_k. The player aims to minimize the functional

$$J = | x_1(T) - y(T) | \tag{1.4}$$

The player Y on the other hand, maximizes the functional (1.4) by realizing the integrable controls $v(t)$ subjected to the restriction (1.3). We shall assume that the player X knows the relations (1.1)—(1.4) and observes the phase vector of the opposing player only at the instants $t_i, i = 1, \ldots, n$, i.e. immediately before the impulse. Consequently the quantities u_k in (1.3) can be regarded as functions of the form $u_k = u_k(x_{1k}, x_{2k}, y_k)$, where x_{1k}, x_{2k} and y_k are the values of the vectors immediately before the k-th impulse. We shall call the controls (1.3) containing such u_k, the strategies of the player X.

Problem 1. To find the optimal guaranteed strategy u^* of the player X and a guaranteed value J^* of the functional (1.4) satisfying the relation

$$J^* = \min_u \sup_v J[u, v] = \sup_v J[u^*, v] \tag{1.5}$$

Here $J[u, v]$ denotes the value of the functional (1.4) corresponding to the data of the strategy u and control v.

2. Equivalent multistep game. Introducing the variable $z(t) = x_1(t) + (T - t) x_2(t) - y(t)$, we can simplify the relations (1.1) and the functional (1.4) as follows:

$$z^{\cdot} = (T - t) u + v, \qquad t \in [0, T]; \qquad J = |z(T)| \qquad (2.1)$$
$$z(0) = z^{\circ} = x_1^{\circ} + Tx_2^{\circ} - y^{\circ}$$

Further, using the technique of reducing the differential games with incomplete information to the equivalent games with complete information (see [4, 5]), we can replace the game (2.1),(1.3) by the game

$$z_{k+1} = z_k + (T - t_k) u_k + (t_{k+1} - t_k) v_k, \qquad z_k = z(t_k - 0) \qquad (2.2)$$
$$z_0 = z^{\circ}, \qquad u_0 = 0, \sum_{k=1}^{n} |u_k| \leqslant Q, \quad |v_k| \leqslant 1, \quad k = 0, \ldots, n, J = |z_{n+1}|$$

The equality $u_0 = 0$ reflects the fact that the player X has no control over his motion up to the instant of the first impulse. Let us introduce a scalar variable q_k representing the remaining resource of the control

$$q_0 = q_1 = Q, \quad q_k = Q - \sum_{i=1}^{k-1} |u_i| \geqslant 0, \quad k = 2, \ldots, n+1 \qquad (2.3)$$

The relation (2.3) yields the following recurrent relation:

$$q_0 = Q, \; q_{k+1} = q_k - |u_k|, \qquad |u_k| \leqslant q_k, \qquad k = 0, \ldots, n \qquad (2.4)$$

Combining (2.2) and (2.4) we obtain the multistep game

$$z_{k+1} = z_k + (T - t_k) u_k + (t_{k+1} - t_k) v_k \qquad (2.5)$$
$$q_{k+1} = q_k - |u_k|, \quad |u_k| \leqslant q_k, \quad |v_k| \leqslant 1, \quad k = 0, \ldots, n$$
$$z_0 = z,^{\circ} \qquad q_0 = Q, \qquad J = |z_{n+1}|$$

To solve the game, we introduce the Bellman function $S_k(z_k, q_k)$ which is equal to the minimum value of the functional J from (2.7) guaranteed for the player X, under the condition that the game begins from the k-th step and the values of phase variables are equal to z_k and q_k. This function satisfies the following recurrence relation with boundary condition (see [5]):

$$S_k(z_k, q_k) = \min_{|u_k| \leqslant q_k} \max_{|v_k| \leqslant 1} S_{k+1}(z_{k+1}, q_{k+1}) \qquad (2.6)$$
$$k = 0, 1, \ldots, n, \qquad S_{n+1}(z_{n+1}, q_{n+1}) = |z_{n+1}|$$

The guaranteed value of the functional (1.5) has the form $J^* = S_0'(z^{\circ}, Q)$. It can be shown by direct verification that the problem (2.6), (2.5) has a unique solution

$$S_k(z_k, q_k) = \max\{f_k^{\,k}, f_{k+1}^{k}, \ldots, f_n^{\,k}, f_{n+1}^{k}\} \qquad (2.7)$$
$$f_m^{\,k} = \vartheta_m \left(\sum_{i=k+1}^{m} \frac{\vartheta_{i-1}}{\vartheta_i} - m + k + 1 + \frac{|z_k| - \vartheta_k q_k}{\vartheta_k} \right)$$
$$m = k, k+1, \ldots, n, \qquad f_{n+1}^{k} = \vartheta_n, \quad k = 1, \ldots, n$$

$$S_0(z^{\circ}, Q) = \max\{f_1^{\circ}, \ldots, f_{n+1}^{\circ}\} = J^* \qquad (2.8)$$
$$f_m^{\circ} = \vartheta_m \left(\sum_{i=1}^{m} \frac{\vartheta_{i-1}}{\vartheta_i} - m + 1 + \frac{|z^{\circ}| - \vartheta_1 Q}{\vartheta_1} \right).$$

$$m = 1, \ldots, n, \ \overset{\circ}{f}_{n+1} = \vartheta_n; \ \vartheta_m = T - t_m, \ m = 0, \ldots, n+1$$

The slight discrepancy between (2.8) and (2.7) can be explained by the fact that the first step in (2.2) is not a standard one ($u_0 = 0$). The quantity f_k°, $k = 1, \ldots, n$ in (2.8) corresponds to the case when the whole resource Q is spent on the first k impulses; the quantity $\overset{\circ}{f}_{n+1}$ corresponds to the case when the resource left after the n-th impulse is not zero.

Computing the extrema in (2.6) yields the following optimal impulses in the game (2.5):

$$u_m^* = -\frac{z_m}{\vartheta_m}, \quad |z_m| \leqslant \vartheta_m q_m; \quad u_m^* = -\frac{z_m}{|z_m|} q_m \tag{2.9}$$

$$|z_m| > \vartheta_m q_m, \quad m = 1, \ldots, n$$

$$v_k^* = \frac{z_k}{|z_k|}, \quad z_k \neq 0; \quad v_k^* = e, \quad |e| = 1, \quad z_k = 0; \quad k = 0, 1, \ldots, n$$

3. Optimization of the impulse times. So far the times of the impulses were assumed fixed. Consider now the following problem.

Problem 2. Assume that the player X knows the quantities z° and Q prior to the beginning of the game. Our aim is to find a distribution t_i^*, $i = 1, \ldots, n$ of the time instances (1.2) for which the quantity (1.5) or, equivalently, (2.8) assumes its minimum value.

It is clear that the required distribution can be found by minimizing the quantity (2.6) over the times $\{t_i\}$ under the restrictions (1.2). Using ϑ_i from (2.8), we can write the restrictions (1.2) in the following form:

$$T = \vartheta_0 \geqslant \vartheta_1 \geqslant \ldots \geqslant \vartheta_n \geqslant \vartheta_{n+1} = 0 \tag{3.1}$$

First we consider the case when

$$|z^\circ| - TQ \geqslant 0 \tag{3.2}$$

From (2.8) it follows that for any distribution of ϑ_i of the form (3.1) we have

$$f_1^\circ \geqslant f_2^\circ \geqslant \ldots \geqslant \overset{\circ}{f}_{n+1}, \qquad J^* = f_1^\circ = |z^\circ| - \vartheta_1 Q + \vartheta_0$$

Thus in the case (3.2) the quantity J^* depends on ϑ_1 only. The optimal value ϑ_1^* of ϑ_1 is found from the condition

$$J^\circ = \min_{0 \leqslant \vartheta_1 \leqslant \vartheta_0} J^* = |z^\circ| + \vartheta_0 (1 - Q) = |z^\circ| + T(1 - Q)$$

and is equal to T, i.e. $t_1^* = t_0 = 0$. Thus when (3.2) holds, the whole resource is used up at the initial instant of time. Now consider the case

$$|z^\circ| - TQ < 0 \tag{3.3}$$

From (2.8) it follows that the relations

$$f_1^\circ \leqslant \ldots \leqslant \overset{\circ}{f}_{k-1} < f_k^\circ > \overset{\circ}{f}_{k+1} > \ldots > \overset{\circ}{f}_{n+1}, \qquad J^* = f_k^\circ \tag{3.4}$$

$$\vartheta_k < \vartheta_{k-1} \tag{3.5}$$

hold for any distribution $\{\vartheta_i\}$ and for certain k, $1 \leqslant k \leqslant n+1$. From (3.4) we find that on the optimal set $\{\vartheta_i^*\}$ although this may not be unique, we can either assume that (3.4) holds for $k = n+1$ or, that

$$f_1^\circ = \ldots = \overset{\circ}{f}_{n-k+1} \leqslant \ldots \leqslant \overset{\circ}{f}_{n-1} < f_n^\circ \geqslant \overset{\circ}{f}_{n+1} \tag{3.6}$$

Indeed, if for $\{\vartheta_i^*\}$ we find that $k \neq n+1$ in (3.4), we construct the set

$\{\vartheta_i'\}$, $\vartheta_i' = \vartheta^*_{k-n+i}$, $i = n-k+1, \ldots, n$, $\vartheta_i' = \vartheta_1^*$, $i = 1, \ldots, n-k$. Using (2.8) and (3.4) we find that the condition (3.6) with the previous value of J° holds on the set $\{\vartheta_i'\}$.

We shall now show that the following equality holds on the optimal set:

$$f_n^\circ = f^\circ_{n+1} = \vartheta_n^* = J^\circ \tag{3.7}$$

Let us assume that the opposite is true: $f_n^\circ \neq f^\circ_{n+1}$. When $f_n^\circ > f^\circ_{n+1}$, (3.6) and (3.5) imply that $\vartheta_n^* < \vartheta^*_{n-1}$. We note that the functions f_k° from (2.8) have the property $\partial f_k^\circ / \partial\vartheta_k < 0$, $k = 1, \ldots, n$. In this case the variation $\delta\vartheta_n > 0$ of the instant ϑ_n^* exists such that the condition (3.6) remains valid on the varied set and the quantity $f_n^\circ = J^\circ$ is strictly decreasing. When $f_n^\circ < f^\circ_{n+1}$, (3.5) yields $\vartheta_{n+1} < \vartheta_n$. Then a variation $\delta\vartheta_n < 0$ of the instant ϑ_n^* can be found such that the quantity $J^\circ = f^\circ_{n+1}$ is strictly decreasing on the varied set. Therefore we can make the quantity f_n° and hence the functional J° strictly decreasing by choosing the quantity $\vartheta_n + \delta\vartheta_n$ in which $\delta\vartheta_n$ is a sufficiently small positive number, as ϑ_n. This however contradicts the initial assumption that J° is a minimum value of the functional.

We therefore conclude that Eq. (3.7) holds on the optimal set $\{\vartheta_i^*\}$. On dividing by ϑ_n, this yields the expression

$$\sum_{i=1}^{n} \frac{\vartheta_{i-1}}{\vartheta_i} - n + \frac{|z^\circ| + \vartheta_1 Q}{\vartheta_1} = 0 \tag{3.8}$$

The equality (3.8) is equivalent to the requirement that the whole resource Q must be used up in n steps. The relation $J^* = \vartheta_n = f_n^\circ$ determines the quantity J^* as the function of $\vartheta_1, \ldots, \vartheta_{n-1}$. The optimal set $\{\vartheta_i^*\}$ represents the extremal point of the function f_n°.

It can easily be verified that when

$$|z^\circ| < TQ/n \tag{3.9}$$

the extremal point is an interior point of the region (3.1). We therefore have

$$\partial f_n^\circ / \partial\vartheta_k = 0, \quad k = 1, \ldots, n-1 \tag{3.10}$$

and this, using (2.8), yields

$$\frac{\vartheta_1}{\vartheta_2} = \frac{|z^\circ| + \vartheta_0}{\vartheta_1}, \quad \frac{\vartheta_{k-1}}{\vartheta_k} = \frac{\vartheta_k}{\vartheta_{k+1}}, \quad k = 2, \ldots, n-1 \tag{3.11}$$

Equations (3.11) together with (3.8) represent n equations from which ϑ_k^* are determined

$$\vartheta_k^* = (|z^\circ| + T)\left(\frac{n}{Q+n}\right)^k \qquad k = 1, \ldots, n \tag{3.12}$$

In the case (3.9), all points (3.12) are die distributed strictly within the interval of motion. If the condition

$$TQ/n \leqslant |z^\circ| < TQ \tag{3.13}$$

is satisfied, then the variable ϑ_1 assumes its limiting value $\vartheta_1^* = \vartheta_0 = T$ at the extremal point of the function f_n° Conditions of the form (3.10) and (3.11) hold for the variables ϑ_k, $k = 2, \ldots, n-1$, and they yield the optimal set for the case (3.13)

$$\vartheta_k^* = T\left(\frac{n-1}{Q - |z^\circ|/T + n - 1}\right)^{k-1}, \quad k = 1, \ldots, n \tag{3.14}$$

The solution constructed above can be utilized for a nonrigorous study of the continuous analog of the game (1.1)—(1.4) in which the player X observes throughout the whole interval $[0, T]$ and applies a finitely impulsive control under the restriction

$$\int_0^T |u(t)|\, dt \leqslant 0$$

From (3.14) it follows that when $n \to \infty$, the instant of application of the last impulse and the value of the functional tend, respectively, to

$$t_\infty = T - \vartheta_\infty, \quad \vartheta_\infty = Te^{|z^\circ|/T-Q}$$

The limiting behavior of the player as $n \to \infty$ supposes the knowledge of the vector $v(t)$ and is as follows. An impulse of intensity $|z^\circ| / T$ is applied at the initial instant, and then the control $u(t) = v(t) / (T - t)$ is applied over the interval $(0, t_\infty]$. This control preserves the zero value of the vector $z(t)$ until all resources have been spent.

We note that the limiting result obtained at $|z^\circ| < TQ$ coincides with the result obtained from the theory developed in [3].

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